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A Study on the Proofs Used by Primary Education Teacher Candidates in Circumference Problem Solutions and Instructional Explanations

Niymet DEMİRCİ[1], Ülkü ÇOBAN SURAL[2], Neşe IŞIK TERTEMİZ[3]

[1] *Gazi University, Ankara, Turkey* *0000-0002-2763-1879*

[2] *Gazi University, Ankara, Turkey* *0000-0002-9766-2443*

[3] *Gazi University, Ankara, Turkey* *0000-0001-7033-7192*

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ABSTRACT

This study aims to determine the proof schemes used by primary education teacher candidates when solving a given problem and in their instructional explanations. Having a qualitative nature, the study utilized data collected from 277 third-year teacher candidates studying at the primary education department of the education faculty of a state university in Ankara. The study group was selected via criterion sampling. The data were tested by using an open-ended problem case and analyzed via document analysis. The proofs used by students in solving the problem and in instructional explanations were categorized as Proof A, Proof B ... Proof H, and the data were evaluated in these categories. The results obtained were discussed in line with the proof schemes outlined by Harel and Sowder (1998). However, as certain proofs fell into several categories, they could not be evaluated in only one group. The primary education teacher candidates were found to use authoritative, habitual and symbolic proof schemes, albeit to a little extent, in the external proof scheme category as they solved the given problem situation and explained the solution to their students. The majority of the candidates used the empirical proof scheme known as the perceptual proof scheme, and included some sample-based proofs. Analytical proof schemes were used less frequently than others. The candidates were also found to use invalid and flawed proofs. Similar studies may be replicated with problem situations involving different subjects.

Keywords:
Mathematics education, proof schemes, teacher candidates, circumference calculation

1. Introduction

Geometry has been one of the main learning areas in all mathematics curricula from past to present due to its benefits in dealing with some daily difficulties that people face, and in the last hundred years, many studies have been made on the effectiveness of teaching and learning processes of geometry, and the nature of geometric thinking and the ways in which it can be developed have been tried to be explained (Güven & Karpuz, 2016). From an early age, students begin to see and understand the physical world around them with the help of geometry. They continue to think high-level geometric thinking in their later education (Kılıç & Tezel-Şahin, 2021).

It is necessary for students to understand geometric concepts and to have knowledge and skills defined in the program in order to be able to use geometry while solving problems. If this situation is not met, geometry is seen by students as a lesson consisting of meaningless features and formulas of shapes (Sezer, 2019). The

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[1]Corresponding author's address: Gazi University, Gazi Faculty of Education, Ankara/Turkey
e-mail: ulkusural@gazi.edu.tr

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properties of geometric shapes and the relations between them are examined in two dimensions: with and without measurements. While the former is known as "dimensional geometry", the latter is known as "non-dimensional geometry" (Altun, 2008). Dimensional geometry includes the topics of perimeter, area, and volume (Tan-Şişman & Aksu, 2009). According to Ayyıldız (2010), although calculating the perimeter is the subject with the least misconceptions among geometry subjects for second graders, there are numerous studies in the literature showing that students have difficulty in learning dimensional geometry (Dağlı & Peker, 2012; Frade, 2005; Gough, 2008; Tan-Şişman & Aksu, 2009; Yeo, 2008). In addition, there are also studies suggesting that area and circumference are two concepts that students have difficulty in understanding and make many mistakes with (Bingölbali & Özmantar, 2014).

The primary responsibility for ensuring that elementary students do not have difficulty in measuring length and do not develop misconceptions undoubtedly falls on primary education teachers who work at the initial stage of education. In order for students to learn length measurement both operationally and conceptually, sufficiently equipped primary education teachers are needed (Tan-Şişman & Aksu, 2009). Many studies have concluded that the most critical element for effective education is teacher knowledge (Hill et al., 2005; Lougran et al., 2004; Toluk-Uçar, 2011). The way the teacher treats subjects and concepts closely affects what and how the learner learns (Bingölbali & Özmantar, 2014). However, an examination of the literature shows that the mathematical understanding obtained by teacher candidates from mathematics courses before and during their university education are insufficient for them to teach at elementary school level (Tirosh, 2000; Toluk-Uçar, 2009). Current studies show that the pre-service teachers know the rules and methods and how they can be applied, but cannot form mathematical explanations suitable for the given situations. In order to be a teacher, candidates must possess content knowledge, field-specific pedagogy knowledge, and knowledge about students' cognitive development (Carpenter et al., 1996). Of these types of knowledge which are part of the extensive information system that a teacher uses when planning and implementing teaching content knowledge and field-specific pedagogy knowledge form the basis of learning (Verschafel et al., 2005).

Studies in the literature show that the instructional explanations used by teachers and teacher candidates are mostly based on memorization without understanding and therefore based on rules and procedures (Zeybek, 2015). However, one of the most important parts of pedagogical knowledge specific to the field of mathematics is to offer instructional explanations in accordance with the rules and concepts related to mathematics. (İlhan & Aslaner, 2020). In addition, considering that pedagogical content knowledge is essentially mathematical knowledge made suitable for teaching, it is obvious that this knowledge is affected by the quality of the teacher's mathematical knowledge. Therefore, information about the instructional explanations provided by teachers on mathematical concepts may enable us to comment on their mathematics knowledge (Risnawati et al., 2019).

Mathematical discoveries and theories involve understanding what is right and what works, explaining why it is right or why it works, and convincing people of them (Weber and Mejia-Ramos, 2019). Proof is the most basic way to show the correctness of a statement and to show the validity of theorems in mathematics (Brown, 2014). According to Sarı-Uzun and Bülbül (2013), proving in the field of mathematics means showing that a conclusion can be logically deduced in line with the given premises, while the resulting valid argument is expressed as proof.

Mathematical proof has a very important role in the development of mathematical thinking and reasoning ability (Stylianides, 2007). The equivalent of understanding another expressive mathematics is proof. (Lee, 2016). That is, proof that provides justification for mathematical knowledge and ensures the internalization of mathematical knowledge is an important factor for doing and understanding mathematics (Buchbinder, 2018). According to Knuth (2002), proof can be used to show and explain the accuracy of something, and to discover and create new mathematical knowledge. Proving is seen as a crucial skill in advanced mathematics, and it is only possible to understand students' proofs by examining their proving processes (Weber, 2001). The "Proof Scheme" framework can be used to effectively analyze conceptual understanding and proof approaches. There are many proof categories created for this purpose (Hanna & Knipping, 2020). The proof scheme concept outlined by Harel and Sowder (1998) explains proof scheme as statements that individuals use to convince themselves or others about the accuracy or inaccuracy of a mathematical situation. It describes the types of proof in three categories and subcategories as expressed in Figure 1.

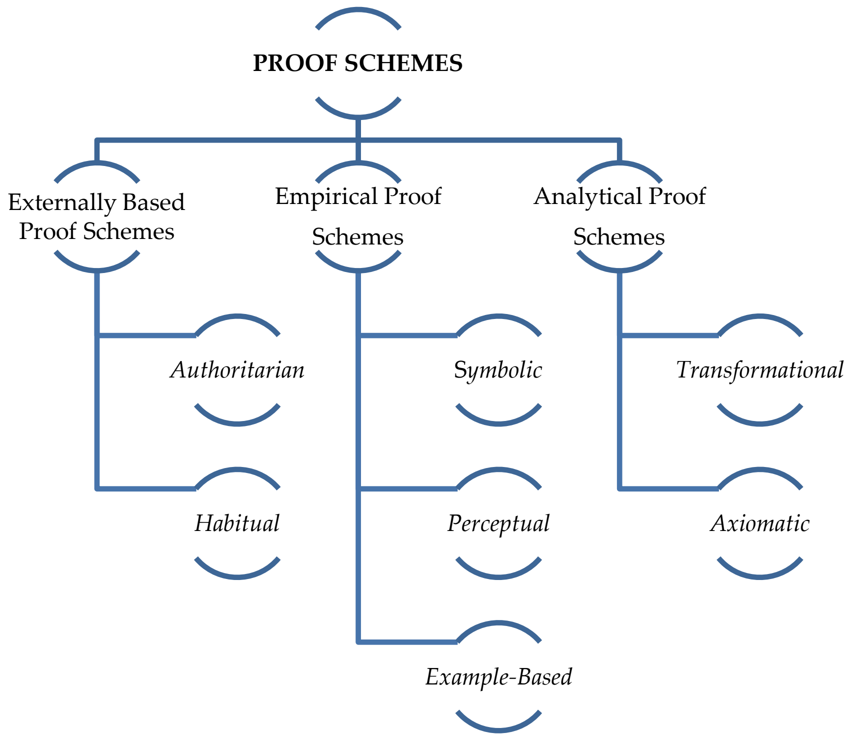


Figure 1. *Proof Schemes and Sub-schemes (cited in Flores, 2006 from Harel & Sowder, 1998).*

Below is a general summary of these schemes:

Externally Based Proof Schemes: In this proof scheme, individuals refer to the people and sources they trust most as the basis for why they know what they learned in mathematics is accurate. In their answers and justifications, textbooks and teachers or adults are used as authority figures (Flores, 2006). In short, students rely on external authority or previously learned proof and reasons to define mathematical validity in the externally based proof scheme (Martin et al., 2005). The externally based proof scheme is divided into three sub-categories: authoritarian, habitual and symbolic (Dede & Karakuş, 2014).

Authoritarian Proof Scheme: In the authoritarian proof scheme, the proof is often based on external teachings. Students explain and prove their solutions and explanations according to the formulas and rules they have memorized. However; they apply the theorems, formulas and rules they have memorized without knowing their meanings(Hanna & Knipping, 2020). In other words, they use rules, definitions or formulas to indicate the correctness of a situation, but they cannot define the rule they use or explain the meaning or starting point of a formula. (Hanna & Knipping, 2020).

Habitual Proof Scheme: In this scheme, students use only the truths they already know instead of reasoning and researching to show the accuracy of a situation. In other words, instead of reasoning, students present proof or justifications they have already learned in order to convince others or themselves (Aydoğdu-İskenderoğlu, 2010). Put differently, the habitual proof scheme involves students remembering the image of a previously proven argument in order to use it in their statements (Harel, 2008).

Symbolic Proof Scheme: In this scheme, students use sequences of arithmetic procedures equal to computational orientation as explanations of their thoughts (Thompson, 1996). Students focus on calculating with the given numbers instead of paying attention to the relationships between the quantities in the situation (Flores, 2002).

Empirical Proof Scheme: This proof scheme includes proof based on specific examples, inductive reasoning or, in the case of geometric proof, direct measurements (Hanna & Knipping, 2020). Flores (2000) explains this proof scheme as basing reasons on perception or appearance of events. Empirical schemes comprise two sub-schemes: perceptual and example-based.

Perceptual Proof Scheme: In this proof scheme, the individual bases the results of a mathematical situation on the perception of a simple drawing, or uses a drawing to persuade the self or others (Aydoğdu-İskenderoğlu, 2010). In the perceptual proof scheme, a hypothesis is validated by primitive mental images or "images composed of perceptions and the coordination of perceptions, but incapable of transforming or predicting the consequences of a transformation" (Harel & Sowder, 1998, p. 255).

Example-Based Proof Scheme: In this proof scheme, students base the accuracy of mathematical expressions on examples and explanations they have learned before (Lee, 2016). In other words, students generally rely on examples when creating concepts. They often use these examples to understand mathematical situations or to check the accuracy of these situations. They use one or more examples to convince themselves or others of the accuracy of an assumption (Flores, 2006).

Analytical Proof Scheme: This proof scheme uses logical deduction and conclusions when demonstrating the accuracy of or validating predictions, assumptions or a mathematical situation. The reasons put forward in the process of ensuring the accuracy and validity of a situation include reasoning as well as being composed of axioms and theorems (Aydoğdu-İskenderoğlu, 2010). Analytical proof schemes are divided into two subcategories: transformational and axiomatic proof schemes (Flores, 2006).

Transformational Proof Scheme: In transformational proof scheme, the individual makes logical inferences based on prior knowledge, realizes relational understanding, reaches generalizations, and convinces others by using deductive and inductive reasoning methods (Dede & Karakuş, 2014). This proof scheme includes students' mental operations completed through deductive reasoning and the expected results of these operations. In other words, it means carrying the characteristics of generalization, operational thinking and logical inference (Aydoğdu-İskenderoğlu, 2016).

Axiomatic Proof Scheme: Axiomatic proof scheme includes all of the features of transformational proof scheme, as well as undefined terms, definitions, predictions, results, theorems and cause-effect relationships in the process of demonstrating the accuracy of a mathematical situation (Lee, 2016). Any student who makes a logical inference via generalization and operational thinking is essentially using the axiomatic proof scheme (Flores, 2006; Harel, 2008; Martin et al., 2005).

Harel and Sowder stated that"these schemes do not exclude one another and students can use more than one proof scheme at the same time" (Harel and Sowder, 1998, p. 244). In a later study, Harel (2008, p. 271) also stated that even though the first two categories, authoritative and empirical proof schemes, are "undesirable ways of thinking", they may still help with "generating ideas or providing insight" thanks to their pedagogical value.

The fact that the subject of proof has recently become the focus of many studies around the world (Çontay & Paksu, 2019; Kosko & Singh, 2019; Lockwood et al., 2020; Nardi & Knuth, 2019) shows that the importance of proof has gradually increased in mathematics education. Despite this, primary, secondary and even higher education students find the proving process difficult and believe that they will fail (Stylianides et al., 2016). This is due to the fact that students have a lack of knowledge about the definitions of proof and how to use it, and that they do not use the mathematical language correctly (Weber & Mejia-Ramos, 2019). In the literature, there are studies showing that teacher candidates also experience similar processes; are insecure about making proofs; cannot understand the proofs of theorems through examination; and that teachers present students with activities that lack the nature of proof and proof-making (Anapa & Şamkar, 2010; Eldekçi, 2018; Köğce, 2013). Indeed, teachers' perceptions and experiences of proof are highly effective in helping students acquire proof skills (Lee, 2016). Therefore, teachers need certain competencies in associating the ideas and representations underlying observable mathematical situations in the classroom (Aslan-Tutak & Köklü, 2016). As a result, understanding teacher candidates' mathematical proof-making skills before they start their careers may enable teacher training institutions to adopt an instructional process that allows teacher candidates to provide more effective instruction to their students in the future.

In addition, revealing the structures that enable teacher candidates to adopt circumference calculation processes may also allow for a deeper analysis of their understanding of the topic of circumference. The literature includes no study conducted in Turkey that deals with the proof schemes used by primary education teachers in their instructional explanations about circumference. For this reason, the main purpose of this study is to determine the proofs used by primary education teacher candidates in solving circumference problems and the proof schemes they use in their instructional explanations. This study seeks answers to the following research questions:

- What are the proofs that primary education teacher candidates use in solving a given circumference problem?
- What are the proofs that primary education teacher candidates use in explaining a given circumference problem to their students?
- What are the errors that primary education teacher candidates make in proving a given circumference problem?

2. Method

2.1. Research Model

This study, which aims to determine the proof schemes used by primary education teacher candidates in the solutions and instructional explanations they offer to circumference problems, was designed according to the basic qualitative research approach. Qualitative research is defined as "research in which qualitative data collection methods such as observation, interview, and document analysis are used and a qualitative process is followed to reveal perceptions and events in their natural environment in a realistic and holistic way" (Yıldırım & Şimşek, 2016, p. 39).

2.2. Study Group

The participants in the study group consisted of 3rd year teacher candidates from the Primary Education undergraduate program of a university in Ankara. The study group was determined by using the purposive sampling method of criterion sampling. In this sampling method, the sample group is formed based on predetermined criteria (Yıldırım & Şimşek, 2016). The criterion used in this study was having completed the courses Elementary Mathematics Education I and Elementary Mathematics Education II. As these courses are offered in year 3, the study was conducted with 3rd year teacher candidates. A total of 277 primary education teacher candidates participated in the study.

2.3. Data Collection Tool

The data collection tools used within the scope of the study consisted of open-ended questions prepared by the researchers. Teacher candidates were asked the following question about measuring circumference, which was written by Dicson et al. (1984) and adapted into Turkish by the researchers. In the selection of the question, effort was made to ensure that it may be answered with all types of proof schemes. The questions mentioned are below;

> *Two rectangles are obtained by dividing a square shaped sheet into two.One of these rectangles is divided into two identical parts from its diagonal. This results in two triangles and a rectangle. These pieces are combined to obtain a parallelogram. Compare the circumference of the initial square and the later parallelogram. Is the square larger than, smaller than or equal to the parallelogram? Write the solution in detail.*
>
> *a. Imagine that you ask this question to your students. One of them responds: "No piece was added. The parallelogram was obtained from the square. Therefore their circumference is still the same". How would you respond to this student? Imagining that you have every material you need for instruction, explain in detail how you would teach this topic to your students in the classroom and how you would prove it.*
>
> *b. How would you convince your students that your proof is accurate? Explain.*
>
> *c. Was this method the first one you thought about?*
>
> *d. Is this method always valid? If possible, give an example of a situation where it is not valid.*
>
> *e. Is there another solution method than the one you have used? Can you use another method for proof? If so, explain.*
>
> *f. Write down the bases of your proof and where/from whom you learned it.*

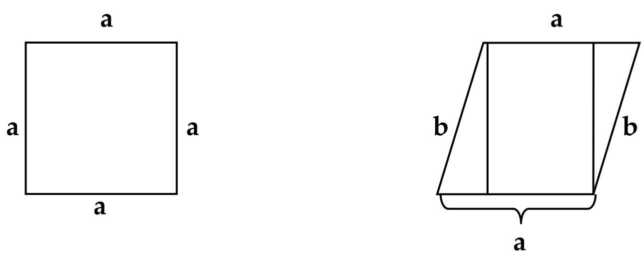
2.4. Data Collection

The data collection tool consisting of open-ended questions was implemented by the researchers on 3rd year students in three different sections of the Primary Education Program in the fall semester of the 2019-2020 academic year. Before the implementation, necessary permissions were obtained from the university and the researchers informed the participants that participation in the study was voluntary, participant identity would be kept confidential, and the data obtained from the study would not be used anywhere else.

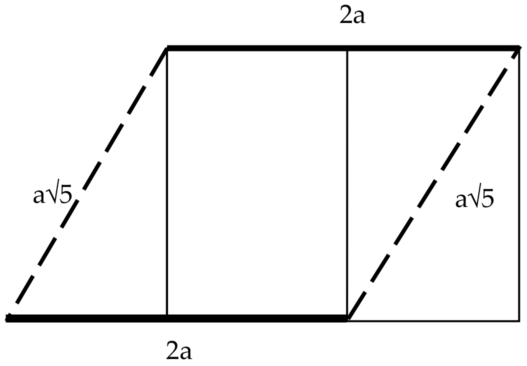
2.5. Data Analysis

The data obtained to reveal the proofs used by primary education teacher candidates in circumference questions and the proofs they used in their instructional explanations were analyzed via content analysis. This method includes objectively and systematically classifying, quantifying and making inferences from the messages contained in verbal, written and other materials by relying on meaning and/or grammar (Yıldırım & Şimşek, 2016). By analyzing the information obtained at this stage, researchers attempt to divide it into meaningful chunks and understand what each part conceptually means. These parts, which are meaningful wholes in themselves, are then named by the researcher (Creswell, 2014). As shown in Table 1, different proofs were determined by the researchers in this study based on the data obtained. These proofs were labelled Proof A, Proof B, ..., Proof H. The codes used while presenting direct quotations about the proofs were a combination of TC meaning "teacher candidate" and the numbers given to the papers. For example, when quoting teacher candidate 3, the code used was TC3.

Table 1. *Proofs Used by Teacher Candidates When Solving Circumference Problems*

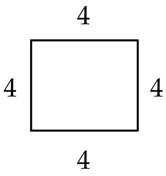
Proofs
Proof A: According to the circumference formula, the circumference of a square is shorter than that of a parallelogram.
Proof B: Let us imagine that the side of a square is 6 cm long, The circumference of the square is: 24 cm If we divide the square into two, we obtain two rectangles with their short sides measuring 3 cm and their long sides measuring 6 cm. If we separate one of these rectangles diagonally, we obtain two right angled triangles, one side measuring 6, and tyhe other side measuring 3 cm. The hypotenuse of the triangle is larger than 6. Therefore, the circumference of the parallelogram is larger than 24 cm.
Proof C:  The circumference of the square is "4a". The circumference of the parallelogram is "2a+2b". In the small triangle formed, the side "b" will be 2b>2a, since the longest side is b>a. Therefore, the circumference of the parallelogram is larger than 2a+2b>4a.
Proof D: When one of two square-shaped pieces of paper of the same size is divided in half as instructed, two rectangles are obtained. When one of the rectangles is cut diagonally, two right angled triangles are obtained, and when these triangles are added to the long side of the rectangle a parallelogram is obtained. When the parallelogram is placed on the square, its circumference can be seen to be longer. The same result is obtained upon repeated trials with squares of different sizes.
Proof E: The circumference of a square piece of paper is measured with a string or ruler. Then the square is converted into a parallelogram as instructed. When the circumference of the parallelogram is measured with a string or ruler and compared to the string used to measure the circumference of the square, the former is longer.

Proof F: It is proved by drawing.



As can be seen, the circumference of the parallelogram formed is longer. Because the longest side of the triangle is a√5.

Proof G: Let's draw a rectangle and a square with the same area on a square sheet of paper.

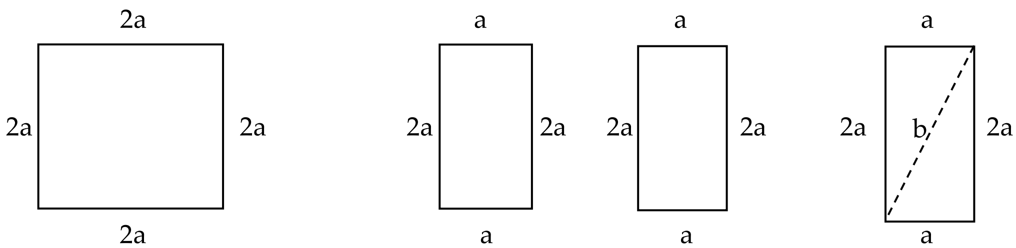


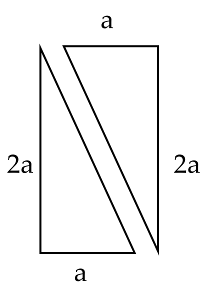
Area of square: 16 unit square

Area of rectangle: 8x2=16 unit square

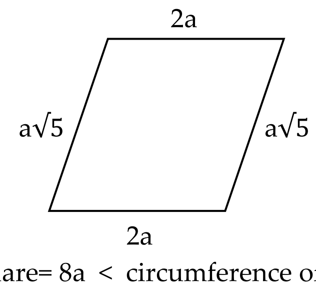
Now let's calculate their circumference. The circumference of the square is 16, while the circumference of the rectangle is 20. The area covered by both shapes on the plane is the same, but their circumference are different.

Proof H:





Using the Pythagorean relation: $b^2 = a^2 + (2a)^2$ $b^2 = 5a^2$



b= a√5 edge is found.

circumference of the square= 8a < circumference of parallelogram= 4a+ 2a√5

The primary education teacher candidates who gave wrong answers to the question were not included in the frequencies. However, the detected errors were analyzed in categories created by the researchers, such as *operational error, area calculation, incomplete solution, inaccurate representation and incorrect inference*.

2.6. Validity and Reliability

Content analysis investigates validity by seeing whether there is harmony between the aims and the tools of research. There is no measurement tool in content analysis other than category definitions. Having clear category definitions informs about what is being measured in the study (Yıldırım & Şimşek, 2016). For these reasons, the proof found in this study was explained in detail and supported by direct examples.

The reliability of the study has been examined by looking at inter-rater reliability and reliability in terms of time. Inter-rater reliability was ensured by looking at the correlation between the results of different researchers. Time-based reliability, on the other hand, was ensured by having the same researchers examine the same documents at a time interval. To calculate correlation, the percentage of agreement between researchers was calculated (Reliability = number of agreements / (number of agreements + number of disagreements)). According to Miles and Huberman (1994), an agreement level above 70% is expected between researchers. The percentage of agreement between researchers in this study was 88%, and the percentage of overlap between the categories showing the causes of detected errors was 92%.

2.7. Ethical

Since this research was conducted before 2020, there is no obligation for an ethics committee decision.

3. Findings

This section presents the findings obtained within the scope of the study. First, the proofs and instructional explanations of the teacher candidates are presented, followed by the errors detected in the solution of the problem.

3.1. Proofs Used by Primary Education Teacher Candidates in Solving Circumference Problems and in Instructional Explanations

The percentages and frequency values of the proofs used by primary education teacher candidates in solving the given circumference problem and in their instructional explanations are presented in Table 2.

Table 2. *Proofs Used by Primary Education Teacher Candidates in Solving Circumference Problems and Instructional Explanations*

Type of Proof	Proofs Used in Problem Solution		Proofs Used in Instructional Explanations	
	f	%	f	%
Proof A	20	8,88	15	6,66
Proof B	20	8,88	30	13,33
Proof C	4	1,77	3	1,33
Proof D	18	8	30	13,33
Proof E	43	19,1	93	43,33
Proof F	41	18.22	17	7,51
Proof G	11	4,88	11	4,88
Proof H	78	34.66	16	7,11
Total	225	100	225	100

As can be seen in Table 2, the proofs used by primary education teacher candidates in solving the given problem may be ordered from the most commonly used to the least as follows: H, E, F, A, B, D, G, D, and C. The proofs used by primary education teacher candidates in explaining the problem to the students may be listed as E, B, D, F, H, A, G, and C, from the most common to the least. In other words, although the proofs used by primary education teacher candidates when solving a problem and the proofs they use when giving explanations to students are the same, their order differs. In further detail, the following can be said:

Proof A: It is the explanation of the accuracy of solutions based on the circumference formula. As shown in Table 2, very few primary education teacher candidates stated, "The circumference formula dictates that the circumference of the square is shorter than that of the parallelogram" when solving the problem (*f* = 20, 8.88%) and when giving instructional explanations (*f* = 15, 6.66%).

Examples:

"I based my proof on circumference formula. ... I learned about this solution method with the circumference formula for the parallelogram and square." (TC 177)

"I did so … by making use of the circumference formula for the parallelogram and square." (TC 57)

"I came across it in an activity book." (TC 230)

"I learned this solution method from the circumference formula for parallelograms and squares." (TC 11)

Proof B: While approximately 10 percent (f = 20, 8.88%) of the primary education teacher candidates used this proof in solving the problem, more than 10 percent (f = 30, 13.33%) preferred this method in their instructional explanations. When explaining the problem situation, the teacher candidates in this group reached the solution by starting from examples and supplying numbers.

Examples:

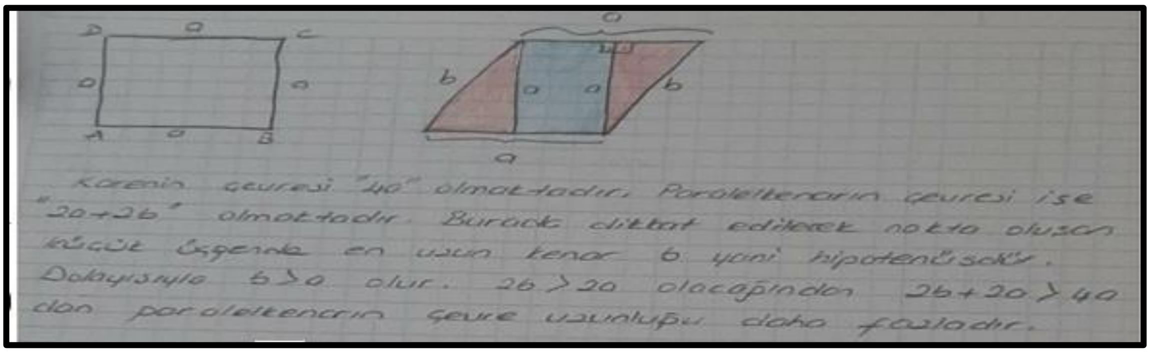
"… A parallelogram is made of a square. So they both cover the same area on a plane. But we cannot say the same about their circumference. … Let's calculate the circumference of these two shapes. The circumference of the square= 6x4 = 24, That of the parallelogram 9+9+4+4 = 26." (TC 14)

"I would give my student a square cardboard with each side 10cm long. I'd ask them to cut it in the middle, make two rectangles and join them horizontally. I'd ask them to compare their circumferences. No additions or spare parts. As they noticed the difference in their circumferences, I'd mention that the circumference changes in the same way in a parallelogram. The circumference of the first square = 40cm, the circumference of the rectangle = 50cm." (TC 65)

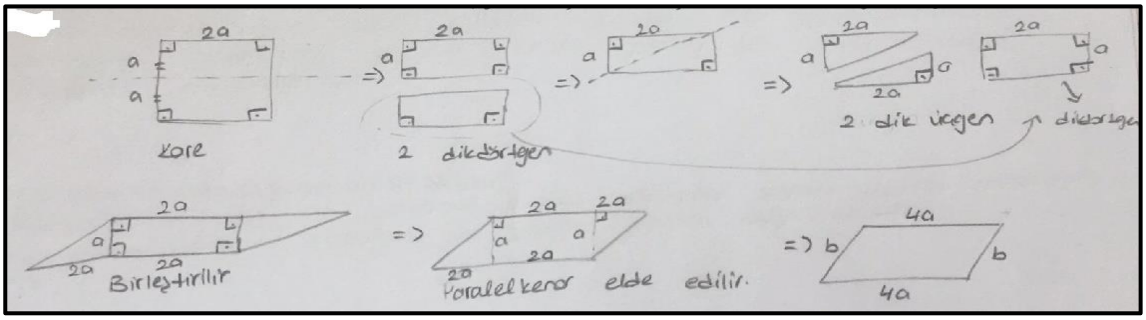
"... For this reason, while 2 sides are a constant" a", the other two sides will be longer than" a". In total, we obtained a parallelogram whose circumference is longer than 4a. As explain this, I distribute squares of 10 cm long sides to my students and let them experience this." (TC 38)

Proof C: This type of proof is the least preferred solution (f = 4, 1.77%) by teacher candidates. This is also true for their instructional explanations (f = 3, 1.33%). The primary education teacher candidates who used Proof C in their explanations based the accuracy of their solutions on examples and giving specific numbers. This can be linked to both authoritative ("b>a" due to the Pythagorean theorem) and symbolic proof schemes.

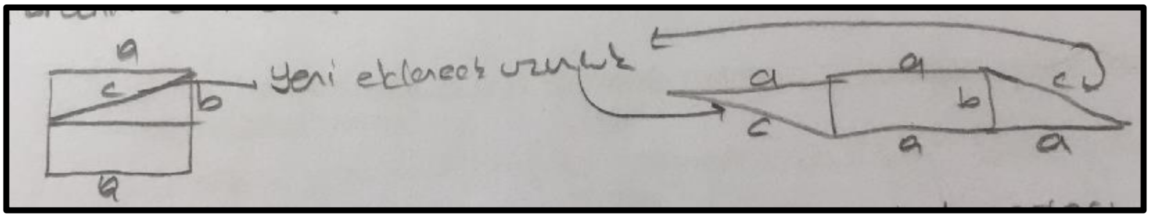
Examples:



TC 104



TC 43



TC 223

Proof D: It is the group that explains the accuracy of their solutions not formally but empirically, via perceptual observations and mental images that are not mature enough. While approximately 10 percent (*f* = 18, 8%) of primary education teacher candidates used this proof in solving the problem, almost 15 percent (*f* = 30, 13%) preferred this method in their instructional explanations.

Examples:

"… I give them a square piece of paper and ask them to do what I did. ... I ask them to do it by giving them a ruler. So I ask them to measure again by turning a square with a known circumference into a parallelogram and compare the measurement made. I repeat this with other square models with different circumference lengths to reinforce it." (TC 14)

"I would create the parallelogram by dividing one of the 2 identical squares as appropriate, and prove that the circumference of the two shapes are not equal when we put them on top of each other. The parallelogram has a longer circumference." (TC 157)

"... I tell the students to take out a graph notebook. I get them to follow the steps above one by one. I let students discover it for themselves. I help them understand that the difference stems from the difference in square and parallelogram lengths." (TC 178)

"… I have my students practice with graph paper. I tell them to draw a square on a graph paper and cut it. I ask them to stick this square on a blank A4 paper. Then I ask them to paint this piece without covering up the squares. I then ask them to draw and cut another identical square. I make them obtain 2 identical right angle triangles by dividing the square into 2 equal parts, horizontally or vertically. Then I ask them to obtain a parallelogram by sticking them next to the square. ... I get them to calculate and compare the circumference units of the square and the parallelogram. With such a practice, I enable them to discover that the circumference of the square is greater than that of the parallelogram." (TC 12)

Proof E: It is the group that explains the accuracy of their solutions by basing them largely on manipulative teaching tools and materials. Approximately 20 percent (*f* = 43, 19%) of primary education teacher candidates used this proof in problem solving, while nearly half (*f* = 93, 43%) used this in their instructional explanations.

Examples:

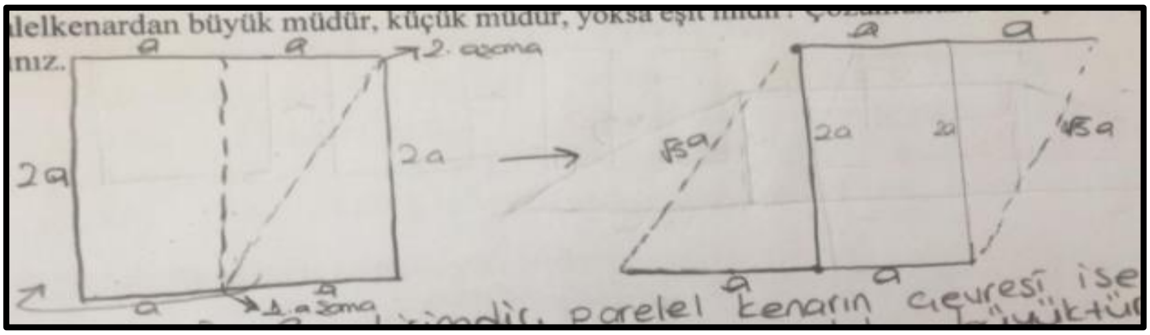
"… I would use string to create a parallelogram and square in class. I would step around the square and then the parallelogram to use step calculation. If the square sides remained the same, the measurements would be equal, but when we take the diagonals, the sides change." (TC 4)

"I distribute two square sheets of equal size to all students and also string that I make sure is longer than the circumference of the square. I ask them not to touch the square I have distributed, and to cut the other square into two equal parts as in the instructions and then to cut one of the rectangles diagonally to obtain two triangles. I then get them to put the pieces together to make a parallelogram on their desks. … I ask them to put the initial squares and the parallelogram side by side, predict their circumferences and take notes. Then I would ask them to cut the string the same length as their circumferences and compare the lengths of the two strings. The string with which they measure the circumference of the parallelogram will be longer. Then I ask them to measure the circumferences with a ruler, make notes and compare them to their predictions." (TC 27)

"... I would get them to draw the shape on colored paper and cut it with the help of scissors. I would ask for it to be measured with a ruler without creating a parallel edge, and I would ask te students to take notes. After creating the parallelogram, I would ask them to measure it again with a ruler to show the difference. In this way, I would reveal that the added piece is longer than one side." (TC 205)

Proof F: In Proof F, the change in the circumference of the parallelogram is explained by analytical-logical inference without using the circumference formula by using the Pythagorean Theorem. While approximately 20% of primary education teacher candidates (*f* = 41, 18%) used this proof in problem solving, almost 8% (*f* = 17, 7,5%) used it in their instructional explanations.

Examples:



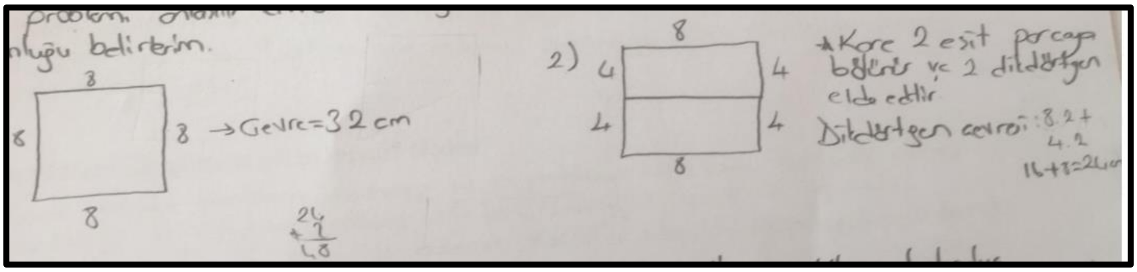
TC 23

"Length increases as you move from right to straight angle. Therefore, the opposite of the right angle in the triangles obtained from the square is longer than the side of the square, and the side of the parallelogram is longer than the square. The circumference of the resulting parallelogram changes and is longer." (TC191)

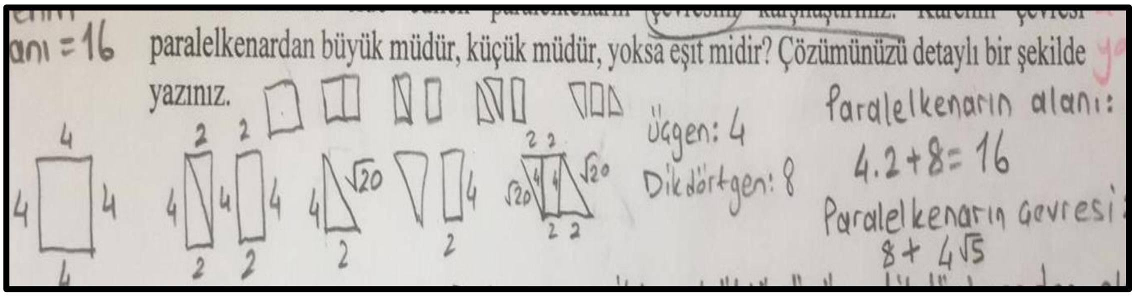
"A line drawn from two corners in rectangles and squares is called a diagonal. These diagonals are longer than the sides. I teach this by getting students to engage in activities with toothpicks and play dough. We make a square out of toothpicks of identical length, and when they place a toothpick of the same length on the two non-adjacent diagonals of this square, they see it is too short and a longer toothpick is needed. When we return to the question after this proof, they understand that the diagonal we drew from the rectangle is longer than the side, and also that the parallelogram is longer than the square."(TC24)

Proof G: Approximately 5% (*f* = 11, 4.8%) of the primary education teacher candidates used this proof both in problem solving and in instructional explanations. The candidates who started from the area-circumference relationship in Proof G and used this as an example made comments such as "*the area covered by the transformed shape on a plane is still the same, but the circumference changes, which also changes the circumference of the parallelogram and makes it longer*".

Examples:



TC 134



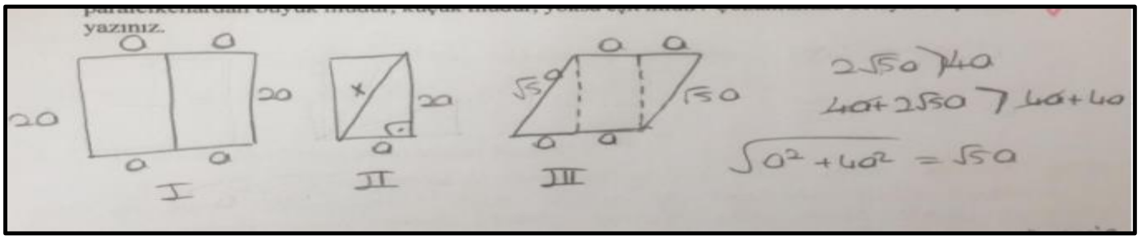
TC 190

"… The circumference of the rectangle is 8.2+2.2=20, and that of the square is 4x4=16. So the circumference of shapes with the same area may be different."(TC27)

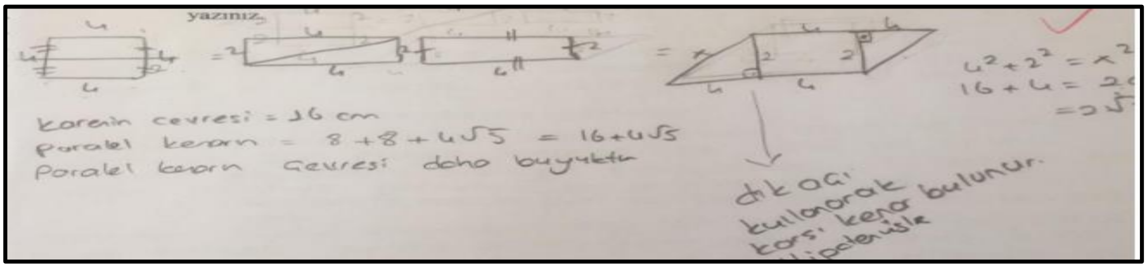
"… the area covered by the transformed shape on a plane is still the same, but the circumference changes, which also changes the circumference of the parallelogram and makes it longer." (TC155)

Proof H: While 34% (*f* = 78, 34%) of the primary education teacher candidates used this proof in problem solving, almost 8% (*f* = 11, 4.8%) used it in their instructional explanations. In Proof H, the Pythagorean theorem and hypotenuse were explained with formulas, while the accuracy of the answers was proven with the circumference formula.

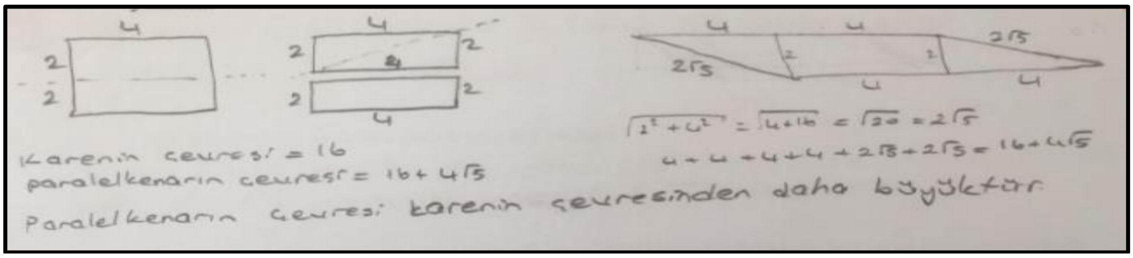
Examples:



TC 8



TC 162



TC 169

Overall, the primary education teacher candidates based the accuracy of their answers on the circumference formula, the Pythagorean theorem and the calculation of the hypotenuse (Proof F (*f* = 41), Proof H (*f* = 78), mean 52%). On the other hand, they mostly based their instructional explanations on perceptual and example-based proof schemes (Proof E (*f* = 93) and Proof D (*f* = 30, mean 59%), or materials, and formal and perceptual expressions.

3.2. Mistakes and Invalid Proofs of Primary Education Teacher Candidates While Proving the Problem Statement Regarding Circumference

Based on the data obtained, it was determined that the errors of primary education teacher candidates who answered the circumference question wrong were caused by *operation error, calculating the area, incomplete solution, incorrect representation and incorrect inference*. These errors are shown in Table 3 with percentages and frequency values. The data are presented with examples from the most common error source to the least common one.

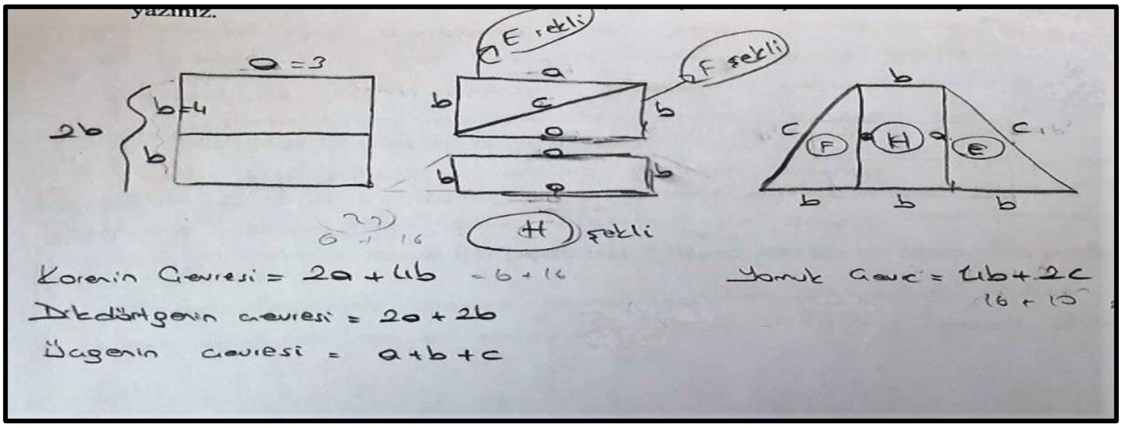
Table 3. *Percentages and Frequency Values of Circumference Calculation Errors by Primary Education Teacher Candidates*

Source of Error	*f*	%
Operation Error	12	20
Calculating are in lieu of circumference	5	8
Wrong inference	23	38
Wrong representation	16	27
Incomplete solution	4	7
Total	60	100

The errors of the primary education teacher candidates in problem solving mostly ($f = 23$, 38%) stemmed from wrong inference. Though the process and the results are correct in the solution, the correct inference is not made.

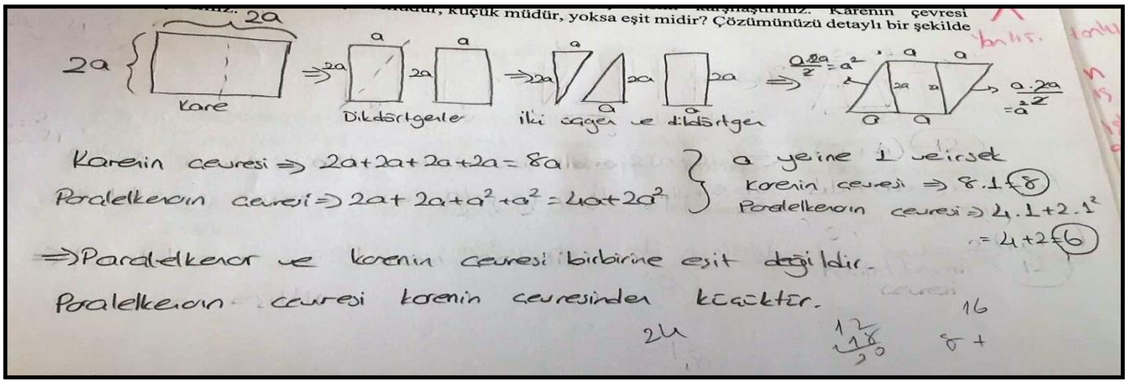
> *"At the beginning, if the circumference of the square is 3 and 5, and the circumference is 2/3+5 =16, when it becomes a parallel side, the short side will be shorter even though one side is the same. Its circumference is smaller than that of the square. It can be solved as approximately 1.5x5. The circumference of the square is greater than the parallelogram."(TC 74)*

In cases of wrong representation ($f = 16$, 27%), formal errors were made as a result of the transformation of the square to a parallelogram.



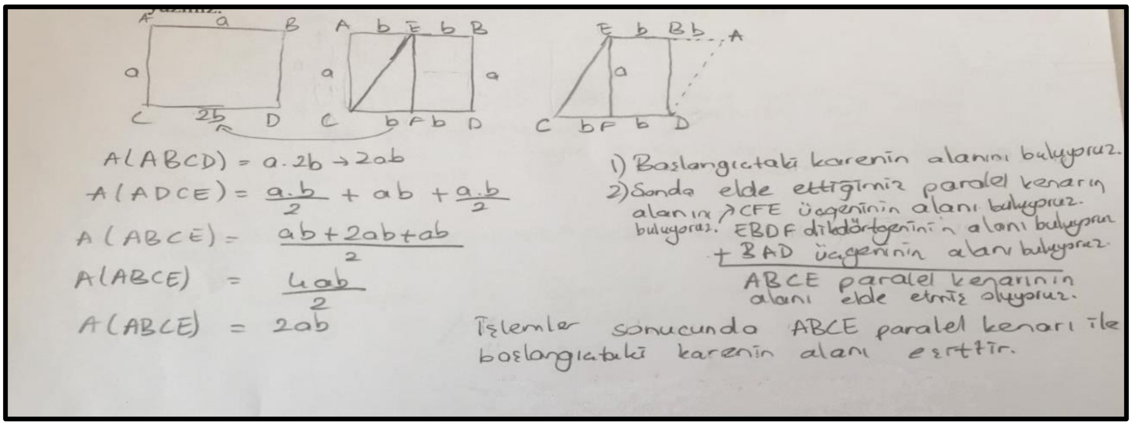
TC 156

Another source of error detected in the solution of the problem is operation error ($f = 12$, 20%). These are mostly mistakes made in calculating the hypotenuse and taking the root.



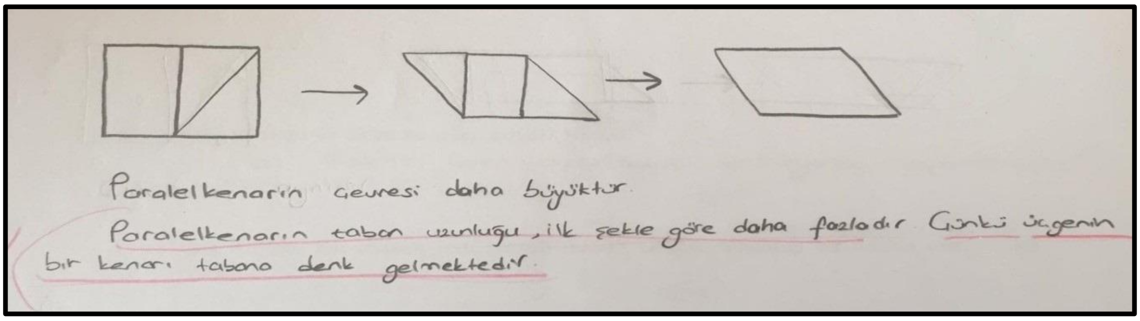
TC 11

There were also wrong solutions caused by calculating area instead of circumference ($f = 5$, 8%). In this source of error, they were found to make inferences by calculating area instead of circumference by using expressions such as "nothing was added to the shape and nothing was reduced from it" during the transformation of square to parallelogram.



TC 143

Primary education teacher candidates made at least (f = 4, 7%) mistakes in solving the problem due to incomplete solution. The problem solving process has not been explained in detail.



TC 4

4. Conclusion and Discussion

The proof schemes used by the primary education teacher candidates in solving a given circumference problem can be ordered from the most common to the least as H, E, F, A, B, D, G, D and C, while the proofs used by them in explaining the solution to the students can be ordered from the most common to the least as E, B, D, F, H, A, G and C. As can be seen, even though the proofs that the teacher candidates used when solving a problem and the proof schemes they used while giving explanations to the students were the same, their order was different. When the proofs used by the candidates are examined in detail, it can be stated that "Proof H" was preferred more (more than one-third) in the solution of the given problem, while one-fifth preferred " Proof F" and " Proof E". However, this changes in favor of "Proof E" in instructional explanations. Nearly half of the candidates preferred " Proof E" to explain the solution of the problem to the students. This was followed by " Proof B" and " Proof D" preferred by one fifth. The proofs may be placed in different categories according to Harel and Sowder's (1998) classification. However, as it is difficult to say that a proof falls into a single category, the same proof can be placed in different categories. In this respect, the study corroborates Housman and Porter's (2003) conclusion in "A Study on the Learning Strategies and Proof Schemes of High Level Mathematics Students" that female students with high-level mathematical thinking skills differ in proving examples. In addition, Raman's (2003) study, in which he examined the proof views of university-level mathematics students and professors, blamed proof problems on university students' lack of mathematical knowledge and skills, and the differences between their ways of proof and those of their professors. This is also an indication that teacher candidates prefer their own proofs in solving a given problem and explaining it to the students.

As they classified proof schemes, Harel and Sowder (1998) emphasized that a proof scheme is a way of thinking and doing mathematics. When the findings obtained in the present study are evaluated in line with this, the following can be stated: Among the proofs used by teacher candidates in solving a given problem and in explaining its solution to the students, authoritative (Proof A), habitual (Proof B, F) and symbolic (Proof C) proof schemes were preferred, albeit to a little extent, in the category of external proof scheme. Proofs from this group showed that in classrooms that can be considered authoritative, as Harel and Sowder (1998) stated in the category of external proof scheme, the proof is based on external authorities such as a teacher or a book rather than a person's own thoughts. When asked "Was this the first method that popped into your mind?", the candidates replied with statements such as "Yes, because this is what I could remember most when it was taught to me", "Yes, owing to the geometry I have learned so far", "From lecture notes", "I saw similar examples in class", "This method is always valid", "This method is regularly used and gives results". Similar to the findings at hand, Aydoğdu-İskenderoğlu (2003) concluded in his study conducted with 5th, 6th, 7th and 8th graders that, among the externally based schemes, the students preferred the authoritative one. Also, Uygan et al. (2014), in a study with pre-service elementary mathematics teachers, similar to the findings, it was determined that pre-service teachers made mistakes in evaluating the reasons that distort the axiomatic structure, in which they tended to external proof more.

As basing geometry solutions on textbooks, teachers and mathematical formulas is a mental habit acquired in education life for primary education teacher candidates who use the externally based proof scheme, their students follow them too (Harel & Sowder, 1998; Aydoğdu-İskenderoğlu, 2016). This habit may also cause primary education teacher candidates to continue their teaching lives with a cycle (Sears, 2012). In addition to these statements, there are studies in mathematics education which argue that most students encounter significant difficulties with proof and that university students use similar proof to high school students (Harel & Sowder, 2007; Stylianous et al., 2009). Pawlikowski (2014) also emphasized in his study that many undergraduate students lack an appropriate reasoning method to ensure the validity of their ideas in geometric proofs.

The most commonly preferred proof scheme among the proofs that primary education teacher candidates use in solving a given problem and in instructional explanations is the empirical proof scheme. It is worth noting that nearly three-quarters of the primary education teacher candidates justify the accuracy of their answers to geometry questions by using the perceptual proof scheme. It was observed that these candidates who used the perceptual proof scheme (Proofs D, E) tended to use materials and manipulative teaching tools in their statements. In the example-based and habitual proof scheme (Proof B), they explained themselves via accepted examples and by providing values. Such explanations given by Harel and Sowder (1998) may be included in the example-based proof scheme subcategory of empirical proof schemes. This often shows that primary education teacher candidates who mostly turn to empirical proof are not equipped enough to effectively handle deductive reasoning (Sears, 2012). Harel and Sowder (1998) state that, in empirical proof schemes, the accuracy of a claim is based on formal reasons (usual forms and appearance) rather than the reasons underlying the proof. In other words, the primary education teacher candidates explained the basis for the accuracy of their solutions with perceptual observations instead of formal explanations. Eldekçi (2018) conducted a study to determine the proof schemes used by pre-service mathematics teachers, and it was determined that pre-service teachers using empirical schemes similarly chose basic examples. In another dimension of the study conducted by Aydoğdu-İskenderoğlu (2003) with 5th, 6th, 7th and 8th graders, it was found that the students who used empirical schemes similarly chose basic examples. Similar results were obtained by Şen and Güler (2015) who contended that secondary school seventh graders generally used empirical proofs. Ünveren (2010), in a part of his study examining the attitudes of elementary school mathematics teacher candidates towards proof, found higher attitude scores in the proofs obtained with mathematical models that can be considered in the perceptual category. Similarly, the fact that teacher candidates mostly use perceptual proof in empirical proof schemes may suggest that this may be due to their attitudes.

In Proof G, the teacher candidates with a different view attempted to prove the problem with the area-circumference relationship. They used an example other than the problem to explain that when two shapes with the same area are transformed, they still cover the same area on a plane but their circumferences have changed. This makes Proof G an example-based proof scheme. Harel and Sowder (1998) generally used the example-based proof scheme to understand the mathematical situations they learned or to check the accuracy of these situations, which further suggests that Proof G is an example-based proof scheme. Flores (2006) also supports this and states that in the example-based proof scheme, students defend used one or more examples to convince themselves or others of the accuracy of an assumption.Stylianides (2007) argues that viewing empirical arguments as proof is a threat to students' opportunities to learn how to prove a proposition. In this case, it can be said that the examples can be qualified as proof in the future classes of teacher candidates.

Another remarkable finding is that only about one-fifth of the primary education teacher candidates used the proof type in the transformation step of the analytical proof scheme category when solving the problem and giving instructional explanations. In this case, Uygan et al. (2014) was also valid in their study with pre-service mathematics teachers Analytical schemes were used less often than others. Even though Proof F and H were evaluated in this group in the present study, the question whether calculating the hypotenuse with the Pythagorean theorem and using the circumference formula may also be evaluated in the externally based proof scheme group comes to mind. On the other hand, Harel and Sowder (1998) define analytical proof schemes as thinking operationally and making logical inferences by using students' inductive and deductive thinking processes. However, primary education teacher candidates who gave instructional explanations using Proof F and H implied that the diagonal of the rectangle was the hypotenuse of the right triangle, and concluded that the circumference of the parallelogram was longer. Logical inference at the end of operational thinking and conceptualizing the hypotenuse suggest that this may be the transformational proof scheme from the analytical proof scheme category. Harel and Sowder’s (2007) arguments that in the transformational proof scheme, variables and other constructed entities may be manipulated; inductive and deductive thinking processes may be used; and generalizations may be possible beyond specific examples support the idea that Proofs F and H are included in transformable proof schemes in instructional explanations. However, these proofs include perceived as well as the expression "a root 5”, thus leading to the question "Does it also fall into the category of habitual proof?"

It can be said that the primary education teacher candidates' lack of enthusiasm to use analytical proof schemes in solving and explaining the given problem may stem from the limited opportunities they had during their secondary education mathematics classes to participate in proving activities, as stated by Harel and Sowder (2007). Emphasizing this in his observations about geometry classes, Hanna (1990) argues, "Proof problems are an educational exercise which often inhibits the mental activity that produces theorems used in proofs." Selden and Selden (2003) also revealed in their study that teacher candidates focused on features other than axiomatic structure and made wrong evaluations while examining a proof. However, teachers' proof concepts and knowledge may be factors that affect their ability to teach proof effectively (Knuth, 2002). The findings obtained in this study regarding analytical proof schemes of primary education teacher candidates corroborate those of previous studies (Eldekçi, 2018; Pawlikowski, 2014; Sears, 2012; Uygan et al., 2014). The proofs used by primary education teacher candidates in solving a given circumference problem and explaining the solution were found to include invalid and wrong proofs. These included operation errors, area calculations, incomplete solutions, wrong representation and errors due to incorrect inferences. Similarly, Weber (2004) also accepted student mistakes as wrong and invalid proofs in a study aiming to explain the processes used by undergraduate students to provide proof.

In sum, it was found that primary education teacher candidates used extrinsic, empirical and analytical proof schemes in solving a given problem and explaining it to the students. However, it is worth noting that the candidates mostly preferred perceptual and example-based proof schemes, which are included in the empirical proof scheme category. Although this is primarily attributed to the traditional proof instruction received by primary education teacher candidates, certain recommendations have still been made to educators and researchers regarding this.

5. Recommendations

Proof, which serves as a tool for learning and teaching mathematics (Hanna, 2000; Knuth, 2002), should become a central goal of mathematics instruction for mathematicians and mathematics teachers (NCTM, 2000). Given the different factors that cause teachers and teacher candidates to resort to low-level cognitive arguments when learning and teaching proof, teacher education programs should aim to support teachers' development of rich proof arguments. For example, task examples that represent cognitive demands at both higher and lower levels may be provided, and teacher candidates may be encouraged to write proofs that necessitate higher cognitive arguments. There is no doubt that teaching practices involving the use of knowledge and proofs in the classroom affect teacher’s teaching potential (Sears, 2012). Therefore, future teachers should be encouraged to see proof as a process that can be used in all aspects of mathematics and to integrate it into their assessment practices (Stylianou et al., 2015). In brief, the development of proof schemes should be included among the teacher competencies aimed by teacher education programs and curricula.

Overall, proof schemes constitute an important tool that help reveal students' misconceptions, ensure more meaningful use of algorithms, and develop students' own thoughts rather than memorization. Therefore, in addition to determining how primary education teacher candidates use proof schemes, how primary education teachers use them may also be examined.

The relationship between primary education teacher candidates' understanding of and beliefs regarding proof may also be investigated. Interviews with teacher candidates may help reveal their thoughts in depth and clarify the data. In addition, the fact that the problem used here was related to measurement and geometry may have caused teacher candidates to resort to empirical proof due to the nature of the subject. Similar future studies may be conducted with problem situations involving different topics.

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